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The spectrum-dependent solutions to the Yang–Baxter equation for quantum E_6 and E_7

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Abstract. In this paper both trigonometric and rational solutions to the spectrum-dependent Yang-Baxter equation are presented for the minimal representations of the quantum exceptional E_6 and E_7 universal enveloping algebras.

1. Introduction

In the statistical vertex models the Boltzmann weight satisfies the spectrum-dependent Yang-Baxter equation. The Ising, tricritical Ising and the tricritical three-state Potts models are described by the level-one cosets $(E_8 \times E_8)/E_8$, $(E_7 \times E_7)/E_7$ and $(E_6 \times E_6)/E_6$, respectively. The solutions to the spectrum-dependent Yang-Baxter equation for the quantum exceptional enveloping algebras have not been well studied. Reshetikhin (1988) discussed the general properties of the solutions without a spectral parameter in terms of the Hopf algebra, and discussed, as an example, the solution for the quantum G_2. We presented the solutions without a spectral parameter for the minimal representations of the quantum E_6, E_7 and F_4 in some detail (Koh and Ma 1990, Kim *et al* 1990). However, the spectrum-dependent solutions are more interesting from the physical viewpoint.

The spectrum-dependent solution for the minimal representation of the quantum G_2 was first discussed by Ogievetsky (1986) and in detail by Kuniba (1990). The most important properties of the spectrum-dependent solutions were discussed by Reshetikhin (1987) and classified by Reshetikhin and Wiegmann (1987). Some results for the solutions were listed by Ogievetsky and Wiegmann (1986) but with some misprints. The rational solutions for the quantum non-exceptional Lie algebras were discussed by Ogievetsky *et al* (1987).

Jimbo raised a principle method for constructing a spectrum-dependent solution (Jimbo 1986). He embedded the deformation of a Kac-Moody universal enveloping algebra (without the central term and without a definition for the coproduct) into the corresponding quantum Lie universal enveloping algebra, and proved that the only solution to the following linear system, if it exists, satisfies the Yang-Baxter equation:

$$[\check{R}_q(x), \Delta(I_\mu)] = 0 \tag{1a}$$

$$(xk_0 \otimes e_0 + e_0 \otimes k_0^{-1})\check{R}_q(x) = \check{R}_q(x)(xe_0 \otimes k_0^{-1} + k_0 \otimes e_0) \tag{1b}$$

where I_μ denotes e_i, f_i or k_i, which span the quantum Lie universal enveloping algebra, and λe_0, $\lambda^{-1} f_0$ and k_0, in addition to I_μ, satisfy the quantum algebraic relations of the

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Kac–Moody universal enveloping algebra. e_0, f_0 and k_0 should be expressed by e_i, f_i and k_i. In terms of this method we obtained the spectrum-dependent solutions for the octet representation of the quantum sl(3) universal enveloping algebra, where there is multiplicity in the decomposition of the coproduct $8\otimes 8$ (Hou *et al* 1990a). Unfortunately, Jimbo did not give the explicit expression of e_0 except for the deformation of $A_n^{(1)}$.

Recently, the quantum algebraic relations of the quantum Lie universal enveloping algebra were realized by a q-analogue of the boson operators (Biedenharn 1989, Sun and Fu 1989, Macfarlane 1989, Hayashi 1990). Song (1990) constructed the explicit expression of e_0 for the deformation of $A_n^{(1)}$ by the realization of the quantum boson operators. It seems to be possible to express e_0 by e_i, f_i and k_i for the deformation of any Kac–Moody universal enveloping algebra.

Our strategy is to make an ansatz that the expression of e_0 as well as the solution $\check{R}_q(x)$ exists, then to compute the spectrum-dependent solution $\check{R}_q(x)$ in terms of the limit conditions, and finally to check whether it satisfies the spectrum-dependent Yang–Baxter solution. In this paper we compute the spectrum-dependent trigonometric solutions for the minimal representations of the quantum E_6 and E_7 universal enveloping algebras (q-E_6 and q-E_7). Then we obtain the rational solutions from the trigonometric ones by an appropriate limit process (Jimbo 1985, Hou *et al* 1990b).

The plan of this paper is as follows. In section 2 we review the properties of the solution $\check{R}_q$ to the Yang–Baxter equation without a spectral parameter for the minimal representation of q-E_6, then construct the spectrum-dependent solution $\check{R}_q(x)$ under the ansatz that both the embedding e_0 in q-E_6 and $\check{R}_q(x)$ exist. In section 3 it will be proved that $\check{R}_q(x)$ does satisfy the spectrum-dependent Yang–Baxter equation. The rational solution for q-E_6 is obtained through an appropriate limit process in section 4. Finally, both trigonometric and rational solutions for the minimal representation of q-E_7 are sketched in section 5.

2. The spectrum-dependent solution for q-E_6

Let r_i and λ_j be the simple roots and the fundamental weights, respectively. r_i and λ_j are related by the Cartan matrix a_{ij}

$$r_i=\sum_j \lambda_j a_{ji} \qquad \lambda_j=\sum_i r_i(a^{-1})_{ij}. \tag{2}$$

For E_6 the Cartan matrix is

$$a=\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & -1\\ 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & -1 & 2 & 0\\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \qquad a^{-1}=\frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3\\ 5 & 10 & 12 & 8 & 4 & 6\\ 6 & 12 & 18 & 12 & 6 & 9\\ 4 & 8 & 12 & 10 & 5 & 6\\ 2 & 4 & 6 & 5 & 4 & 3\\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix} \tag{3}$$

and the Dynkin diagram is shown in figure 1.

The states in an irreducible representation N are denoted by their weights m, and the highest weight m is equal to N. Both N and m are the integral combinations of λ_j:

$$N=\sum_j (N)_j\lambda_j \qquad m=\sum_j (m)_j\lambda_j. \tag{4}$$

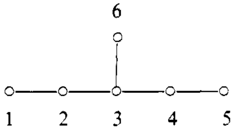

Figure 1. Dynkin diagram of E_6.

The minimal representatiaon $N_0 = \lambda_1$ is denoted by (100000). The decomposition of the tensor product of N_0 for E_6 is given as follows

$$(100000) \otimes (100000) = (200000) \oplus (010000) \oplus (000010) \tag{5a}$$

or briefly

$$N_0 \otimes N_0 = N_1 \oplus N_2 \oplus N_0^* \tag{5b}$$

where $N_1 = 2\lambda_1$, $N_2 = \lambda_2$ and $N_0^* = \lambda_5$. The explicit form of the solution $\check{R}_q$ without a spectral parameter for the representation N_0 of q-E_6 was given by Koh and Ma 1990†

$$\check{R}_q = \sum_N \Lambda_N(q)(C_q)_N(\tilde{C}_q)_N \tag{6}$$

$$\check{R}_q^{-1} = \sum_N \Lambda_N(q^{-1})(C_q)_N(\tilde{C}_q)_N \tag{7}$$

where $N = N_1$, N_2 and N_0^*,

$$\Lambda_N(q) = \xi_N q^{C_2(N_1) - C_2(N)} \tag{8}$$

$$\begin{aligned} &\xi_{N_1} = -\xi_{N_2} = \xi_{N_0^*} = 1 \\ &C_2(N_1) = \tfrac{56}{3} \qquad C_2(N_2) = \tfrac{50}{3} \qquad C_2(N_0) = C_2(N_0^*) = \tfrac{26}{3} \end{aligned} \tag{9}$$

$$\Lambda_{N_1}(q) = 1 \qquad \Lambda_{N_2}(q) = -q^2 \qquad \Lambda_{N_0^*}(q) = q^{10} \tag{10}$$

and $(C_q)_N$ denotes the quantum Clebsch–Gordan matrix which reduces the coproduct $N_0 \otimes N_0$ onto the representation N.

Now we return to (1) for the spectrum-dependent solution $\check{R}_q(x)$. We assume that the deformation of the Kac–Moody $E_6^{(1)}$ can be embedded into q-E_6, i.e. the generator e_0, f_0 and k_0 can be expressed by the generators e_i, f_i and k_i of q-E_6. All the generators of the deformation of $E_6^{(1)}$ $(i = 0, 1, 2, \ldots, 6)$ satisfy the standard quantum algebraic relations (Jimbo 1985). It is worth emphasizing that

$$[e_0, f_i] = 0 \qquad i = 1, 2, \ldots, 6. \tag{11}$$

Note that there is no definition for the coproduct of e_0 and f_0. When $q = 1$, e_0 is nothing but the generator corresponding to the lowest negative root r_0 in E_6:

$$\begin{aligned} r_0 &= -\lambda_6 \\ &= \sum_j \alpha_j r_j \\ &= -r_1 - 2r_2 - 3r_3 - 2r_4 - r_5 - 2r_6. \end{aligned} \tag{12}$$

† In this paper we use a slightly different notation from that in our previous paper (Koh and Ma 1990). We remove a factor $q^{-4/3}$ in $\check{R}$, i.e. $q^{4/3}\check{R}$ there is now denoted by $\check{R}_q$. The Casimir operator $C_2(N)$ is divided by two. A term $-q^3(q^2-1)(E_{89} \otimes E_{43} + \text{'transpose'})$ was dropped in (9c) of Koh and Ma (1990).

Furthermore, we assume that the solution $\check{R}_q(x)$ to the linear system (1) exists. From (1a), $\check{R}_q(x)$ is a combination of the projectors owing to the Schur theorem:

$$\check{R}_q(x)=\sum_N \Lambda_N(x,q)(C_q)_N(\tilde{C}_q)_N \tag{13}$$

where $N=N_1$, N_2 and N_0^*. Since the solution $\check{R}_q(x)$ of (1) is unique up to a coefficient (Jimbo 1986), it is easy to see from (1) that $\check{R}_q(x)$ goes to a constant matrix when x goes to 1:

$$\check{R}_q(1)=c\mathbb{1}. \tag{14}$$

Akutsu and Wadati (1987) showed that

$$\lim_{x\to 0}\check{R}_q(x)\sim\check{R}_q \qquad \lim_{x\to\infty}\check{R}_q(x)\sim\check{R}_q^{-1}. \tag{15}$$

Choosing the coefficient in $\check{R}_q(x)$, we have

$$\check{R}_q(0)=\check{R}_q \qquad \Lambda_N(0,q)=\Lambda_N(q). \tag{16}$$

$\Lambda_N(x,q)$ will be determined by (1b), which is a matrix equation operating on the states $|N_0m_1\rangle|N_0m_2\rangle$. Transforming the states into the states $|Nm\rangle$ of the irreducible representation N, we obtain from (1b)

$$\{xX(q)_{N'm',Nm}+Y(q)_{N'm',Nm}\}\Lambda_N(x,q)=\Lambda_{N'}(x,q)\{X(q)_{N'm',Nm}+xY(q)_{N'm',Nm}\} \tag{17}$$

where

$$\begin{aligned}X(q)_{N'm',Nm}&=\sum_{\substack{m_1m_2\\m_1'm_2'}}(C_q)_{m_1'm_2'N'm'}D_q^{N_0}(k_0)_{m_1'm_1}D_q^{N_0}(e_0)_{m_2'm_2}(C_q)_{m_1m_2Nm}\\ Y(q)_{N'm',Nm}&=\sum_{\substack{m_1m_2\\m_1'm_2'}}(C_q)_{m_1'm_2'N'm'}D_q^{N_0}(e_0)_{m_1'm_1}D_q^{N_0}(k_0^{-1})_{m_2'm_2}(C_q)_{m_1m_2Nm}\end{aligned} \tag{18}$$

and $D_q^{N_0}(k_0)$ and $D_q^{N_0}(e_0)$ are the representation matrices of k_0 and e_0 in the irreducible representation N_0. Since e_0 corresponds to the lowest negative root r_0 (see (12)), we have

$$m'=m+r_0. \tag{19}$$

$\Lambda_N(x,q)$ should be independent of m' as well as m, so we can set m to be the highest weight, $m=N$, when we compute the ratio $\Lambda_N(x,q)/\Lambda_{N'}(x,q)$. Generally, equation (17) is overdetermined for $\Lambda_N(x,q)$. The ansatz of the existence of $\check{R}_q(x)$ means that equation (17) is consistent. We are going to show that both $e_0\otimes k_0^{-1}$ and $k_0\otimes e_0$ are commutable with the coproduct $\Delta(f_i)=f_i\otimes k_i^{-1}+k_i\otimes f_i$, $i\neq 0$, so that the ratio $\Lambda_N(x,q)/\Lambda_{N'}(x,q)$ computed from (17) with the lower m is the same as that with the highest m, or the ratio computed with the higher m' is the same as that with the lowest m'.

It is obvious that $k_0\otimes e_0$ is commutable with $k_i\otimes f_i$ due to (11). From the quantum algebraic relations of the Kac-Moody universal enveloping algebra (Jimbo 1985) we have

$$(k_0\otimes e_0)(f_i\otimes k_i^{-1})=(f_i\otimes k_i^{-1})(k_0\otimes e_0)q^{\Sigma_j(-b_{ji}+b_{ij})\alpha_j}$$

where $b_{ji}=b_{ij}=\frac{1}{2}(r_j,r_j)a_{ji}=(r_j,r_i)$ is the symmetric Cartan matrix, and α_j is the component of r_0. Therefore, $k_0\otimes e_0$ is commutable with $\Delta(f_i)$. The proof for the commutability of $e_0\otimes k_0^{-1}$ with $\Delta(f_i)$ is similar.

Now, we return to (17) where $m=N$ and N is higher than N'. Equation (17) is trivial if $m'=N+r_0$ does not belong to the representation N'. In our problem, $m'=N_1+r_0=2\lambda_1-\lambda_6$ belongs to N_2 but not to N_0^*, and $m''=N_2+r_0=\lambda_2-\lambda_6$ belongs to N_0^*. Therefore, we only have two independent relations for the $\Lambda_N(x,q)$

$$\begin{aligned}\frac{\Lambda_{N_1}(x,q)}{\Lambda_{N_2}(x,q)}&=\frac{X(q)_{N_2(N_1+r_0),N_1N_1}+xY(q)_{N_2(N_1+r_0),N_1N_1}}{xX(q)_{N_2(N_1+r_0),N_1N_1}+Y(q)_{N_2(N_1+r_0),N_1N_1}}\\ \frac{\Lambda_{N_2}(x,q)}{\Lambda_{N_0^*}(x,q)}&=\frac{X(q)_{N_0^*(N_2+r_0),N_2N_2}+xY(q)_{N_0^*(N_2+r_0),N_2N_2}}{xX(q)_{N_0^*(N_2+r_0),N_2N_2}+Y(q)_{N_0^*(N_2+r_0),N_2N_2}}.\end{aligned}\tag{20}$$

It is easy to obtain from the limit conditions (16) and (10) that

$$\begin{aligned}\Lambda_{N_1}(x,q)&=(1-xq^2)(1-xq^8)\\ \Lambda_{N_2}(x,q)&=(x-q^2)(1-xq^8)\\ \Lambda_{N_0^*}(x,q)&=(x-q^2)(x-q^8)\end{aligned}\tag{21}$$

Obviously,

$$\begin{aligned}&\check{R}_q(x)=\widetilde{\check{R}_q(x)}\\ &\check{R}_q(1)=(1-q^2)(1-q^8)\mathbb{1}\\ &\check{R}_q(x)\check{R}_q(x^{-1})=x^{-2}(1-xq^2)(x-q^2)(1-xq^8)(x-q^8)\mathbb{1}.\end{aligned}\tag{22}$$

Substituting (21) and (22) into (13), we have

$$\check{R}_q(x)=\check{R}_q+x\{(1-q^2)(1-q^8)\mathbb{1}-\check{R}_q-q^{10}\check{R}_q^{-1}\}+x^2q^{10}\check{R}_q^{-1}.\tag{23}$$

Since

$$\check{R}_q^{-1}=P\check{R}_{q^{-1}}P$$

where P denotes the transposition: $P\in\mathrm{End}(V_1\otimes V_2)$, $P(u\otimes v)=v\otimes u$, we have

$$q^{-5}x^{-1}\check{R}_q(x)=P\{q^5x\check{R}_{q^{-1}}(x^{-1})\}P.\tag{24}$$

3. The spectrum-dependent Yang–Baxter equation

We do not have the exact expression of e_0 in terms of e_i, f_i and h_i, so it is necessary to check whether or not $\check{R}_q(x)$ in (23) satisfies the spectrum-dependent Yang–Baxter equation

$$\check{R}_q^{12}(x)\check{R}_q^{23}(xy)\check{R}_q^{12}(y)=\check{R}_q^{23}(y)\check{R}_q^{12}(xy)\check{R}_q^{23}(x).\tag{25}$$

Equation (25) is a polynomial in x and y. Since $\check{R}_q$ is a solution to the Yang–Baxter equation without a spectral parameter x:

$$\check{R}_q^{12}\check{R}_q^{23}\check{R}_q^{12}=\check{R}_q^{23}\check{R}_q^{12}\check{R}_q^{23}\tag{26}$$

it is easy to check that (25) is satisfied in the following limit cases: (i) $|x|\gg1$; (ii) $|x|\ll1$; (iii) $|y|\gg1$; (iv) $|y|\ll1$; (v) $x=1$; (vi) $y=1$; (vii) the term of x^2y^2 of both sides of (25) are equal to each other. Furthermore, from (5) and (8) we have the reduction relation for $\check{R}_q$

$$(\check{R}_q-1)(\check{R}_q+q^2)(\check{R}_q-q^{10})=0.\tag{27}$$

The coefficients of both the term xy^3 and x^3y are satisfied, respectively, owing to (16) and (27). Now, the spectrum-dependent Yang-Baxter equation (25) reduces to the following relation for the coefficients of the term xy:

$$\begin{aligned} &\check{R}_q^{12}S_q^{23}\check{R}_q^{12}+S_q^{12}\check{R}_q^{23}S_q^{12}-\check{R}_q^{23}S_q^{12}\check{R}_q^{23}-S_1^{23}\check{R}_q^{12}S_q^{23}=0 \\ &S_q=(1-q^2)(1-q^8)\mathbb{1}-\check{R}_q-q^{10}\check{R}_q^{-1}. \end{aligned} \tag{28}$$

In order to prove (28), we borrow the new idea of Wang (1990) to introduce the Birman-Wenzl-Murakami algebra (Deguchi *et al* 1988). Let

$$\begin{aligned} &G_i=\mathbb{1}^{(1)}\otimes\mathbb{1}^{(2)}\otimes\ldots\otimes\mathbb{1}^{(i-1)}\otimes\check{R}_q\otimes\mathbb{1}^{(i+2)}\otimes\ldots\otimes\mathbb{1}^{(n)} \\ &E_i=\mathbb{1}^{(1)}\otimes\ldots\otimes\mathbb{1}^{(i-1)}\otimes(\check{R}_q-1)(\check{R}_q+q^2)\otimes\mathbb{1}^{(i+2)}\otimes\ldots\otimes\mathbb{1}^{(n)}. \end{aligned} \tag{29}$$

Obviously, we have the following relations:

$$(G_i-1)(G_i+q^2)(G_i-q^{10})=0 \tag{30a}$$

$$E_i=(G_i-1)(G_i+q^2) \tag{30b}$$

$$E_i^2=(q^{10}-1)(q^{10}+q^2)E_i \tag{30c}$$

$$E_iG_i=G_iE_i=q^{10}E_i \tag{30d}$$

$$G_iG_{i\pm1}G_i=G_{i\pm1}G_iG_{i\pm1} \tag{30e}$$

$$G_iG_j=G_jG_i \qquad \text{for } |i-j|\geqslant 2 \tag{30f}$$

$$E_iE_j=E_jE_i \qquad \text{for } |i-j|\geqslant 2. \tag{30g}$$

From (6) and (10) we have

$$(\check{R}_q-1)(\check{R}_q+q^2)=(q^{10}-1)(q^{10}+q^2)(C_q)_{N_0^*}(\tilde{C}_q)_{N_0^*}. \tag{31}$$

In terms of the explicit forms of $\check{R}_q$ and $(C_q)_{N_0^*}$ we can show that

$$E_iE_{i\pm1}E_i=q^{20}(1-q^2)^2E_i \tag{30h}$$

$$E_iG_{i\pm1}G_i=-\frac{q^{-8}}{(1-q^2)}E_iE_{i\pm1}. \tag{30i}$$

Then it is easy to show G_i and E_i satisfy the rest of the relations of the modified Birman-Wenzl-Murakami algebra:

$$G_{i\pm1}E_iG_{i\pm1}=q^4G_i^{-1}E_{i\pm1}G_i^{-1} \tag{30j}$$

$$G_{i\pm1}E_iE_{i\pm1}=-q^{12}(1-q^2)G_i^{-1}E_{i\pm1} \tag{30k}$$

$$E_{i\pm1}E_iG_{i\pm1}=-q^{12}(1-q^2)E_{i\pm1}G_i^{-1} \tag{30l}$$

$$E_iG_{i\pm1}E_i=-q^2(1-q^2)E_i. \tag{30m}$$

By making use of (30) we obtain

$$\begin{aligned} &G_{i\pm1}^{-1}G_iG_{i\pm1}^{-1}-G_i^{-1}G_{i\pm1}G_i^{-1} \\ &\quad=(1-q^2)\{(q^{-2}-q^{-4}+q^{-12})(G_i-G_{i\pm1})-q^{-10}(G_i^{-1}-G_{i\pm1}^{-1}) \\ &\quad\quad-q^{-2}(G_iG_{i\pm1}^{-1}+G_{i\pm1}^{-1}G_i-G_i^{-1}G_{i\pm1}-G_{i\pm1}G_i^{-1})\} \\ &G_iG_{i\pm1}^{-1}G_i-G_{i\pm1}G_i^{-1}G_{i\pm1} \\ &\quad=(1-q^2)\{-q^8(G_i-G_{i\pm1})+(1-q^2+q^{10})(G_i^{-1}-G_{i\pm1}^{-1}) \\ &\quad\quad+(G_iG_{i\pm1}^{-1}+G_{i\pm1}^{-1}G_i-G_i^{-1}G_{i\pm1}-G_{i\pm1}G_i^{-1})\}. \end{aligned} \tag{32}$$

Now, equation (28) is satisfied through the straightforward calculation.

4. Rational solution for q-E$_6$

Through an appropriate limit process, a rational solution $\check{R}(u, \eta)$ can be obtained from a spectrum-dependent trigonometric solution $\check{R}_q(x)$ (Jimbo 1985, Hou *et al* 1990b). Let $x = q^{2u/\eta}$ and take the limit $q \to 1$; we obtain

$$\begin{aligned}\check{R}(u, \eta) &= \lim_{q\to 1} \check{R}_q(q^{2u/\eta})/(1-q^{2u/\eta})^2 \\ &= (1+5\eta/u+4\eta^2/u^2)C_{N_1}\tilde{C}_{N_1} + (-1-3\eta/u+4\eta^2/u^2)C_{N_2}\tilde{C}_{N_2} \\ &\quad + (1-5\eta/u+4\eta^2/u^2)C_{N_0^*}\tilde{C}_{N_0^*} \\ &= P\{\mathbb{1} + (\eta/u)(2t+\tfrac{11}{3}\mathbb{1}) + 4(\eta^2/u^2)P\}\end{aligned} \tag{33}$$

$$t = \sum_a I_a \otimes I_a \tag{34}$$

where I_a denotes the orthogonal bases of E$_6$, and the CG matrix C_N for E$_6$ is the eigenstate of t

$$C_N = (C_q)_N|_{q=1} \qquad tC_N = \tfrac{1}{2}\{C_2(N) - 2C_2(N_0)\}C_N.$$

The rational solution $\check{R}(u, \eta)$ satisfies the general property discussed at the end of Hou *et al* (1990b).

5. The spectrum-dependent solution for q-E$_7$

The method of constructing the spectrum-dependent solutions to the Yang-Baxter equation given in section 2 can be easily generalized to the other Lie universal enveloping algebras. As an example, we use it to construct the spectrum-dependent solutions, both trigonometric and rational, to the Yang-Baxter equation for the minimal representation (0000010) of q-E$_7$. The decomposition of the tensor product of E$_7$ is as follows:

$$(0000010)\otimes(0000010) = (0000020)\oplus(0000100)\oplus(1000000)\oplus(0000000) \tag{35a}$$

or briefly,

$$N_0 \otimes N_0 = N_1 \oplus N_2 \oplus N_3 \oplus N_4 \tag{35b}$$

where $N_0 = \lambda_6$, $N_1 = 2\lambda_6$, $N_2 = \lambda_5$, $N_3 = \lambda_1$ and $N_4 = 0$. The Dynkin diagram of E$_7$ is shown in figure 2.

The solution $\check{R}_q$ to the Yang-Baxter equation without a spectral parameter is (Koh and Ma 1990)

$$\check{R}_q = \sum_{i=1}^{4} \Lambda_i(q)(C_q)_{N_i}(\tilde{C}_q)_{N_i} \tag{36}$$

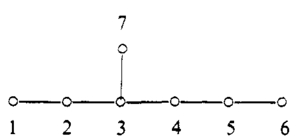

Figure 2. Dynkin diagram of E$_7$.

where

$$\begin{aligned}
&\Lambda_i(q)=\xi_i q^{C_2(N_1)-C_2(N_i)}\\
&\xi_1=-\xi_2=\xi_3=-\xi_4=1\\
&C_2(N_1)=30 \qquad C_2(N_2)=28 \qquad C_2(N_3)=18 \qquad C_2(N_4)=0\\
&\Lambda_1(q)=1 \qquad \Lambda_2(q)=-q^2 \qquad \Lambda_3(q)=q^{12} \qquad \Lambda_4(q)=-q^{30}.
\end{aligned} \tag{37}$$

The spectrum-dependent solution of the Yang-Baxter equation can be expressed as

$$\check{R}_q(x)=\sum_{i=1}^{4}\Lambda_i(x,q)(C_q)_{N_i}(\tilde{C}_q)_{N_i} \tag{38}$$

with the limit conditions

$$\check{R}_q(0)=\check{R}_q \qquad \Lambda_i(0,q)=\Lambda_i(q). \tag{39}$$

The lowest negative root of E_7 is $r_0=-\lambda_1$. Because $N_1+r_0=2\lambda_6-\lambda_1$, $N_2+r_0=\lambda_5-\lambda_1$ and $N_3+r_0=0$ are the Weyl reflections of the dominant weights N_2, N_3, N_4, respectively, the only constraints to the ratio of $\Lambda_i(x,q)$ are

$$\frac{\Lambda_i(x,q)}{\Lambda_{i+1}(x,q)}=\frac{X(q)_{N_{i+1}(N_i+r_0),N_iN_i}+xY(q)_{N_{i+1}(N_i+r_0),N_iN_i}}{xX(q)_{N_{i+1}(N_i+r_0),N_iN_i}+Y(q)_{N_{i+1}(N_i+r_0),N_iN_i}} \qquad i=1,2,3.$$

Therefore, we have

$$\begin{aligned}
\Lambda_1(x,q)&=(1-xq^2)(1-xq^{10})(1-xq^{18})\\
\Lambda_2(x,q)&=(x-q^2)(1-xq^{10})(1-xq^{18})\\
\Lambda_3(x,q)&=(x-q^2)(x-q^{10})(1-xq^{18})\\
\Lambda_4(x,q)&=(x-q^2)(x-q^{10})(x-q^{18}).
\end{aligned} \tag{40}$$

Obviously, $\check{R}_q(x)$ have the following properties:

$$\begin{aligned}
&\check{R}_q(x)=\widetilde{\check{R}_q(x)} \qquad \check{R}_q(1)=(1-q^2)(1-q^{10})(1-q^{18})\mathbb{1}\\
&\check{R}_q(0)=\check{R}_q \qquad x^{-3}\check{R}_q(x)|_{x\sim\infty}=-q^{30}\check{R}_q^{-1}\\
&\check{R}_q(x)=P\{-q^{30}x^3\check{R}_{q^{-1}}(x^{-1})\}P\\
&\check{R}_q(x)\check{R}_q(x^{-1})=x^{-3}(1-xq^2)(x-q^2)(1-xq^{10})(x-q^{10})(1-xq^{18})(x-q^{18})\mathbb{1}.
\end{aligned} \tag{41}$$

Note that the representation N_4 is the trivial one ($N_4=0$), so we have

$$(C_q)_{m_1m_200}=\delta_{m_1\bar{m}_2}\frac{(-1)^{\kappa(m_1)}}{d_q(N_0)^{1/2}}q^{-2\rho(m_1)} \tag{42}$$

where $\bar{m}_2=-m_2, \kappa(m_1)=\Sigma_i p_i$ if $N_0-m_1=\Sigma_i p_i r_i$, $\rho(m)=\frac{1}{2}\Sigma_{ij}(a^{-1})_{ji}(m)_i$ if $m=\Sigma_i(m)_i\lambda_i$, and $d_q(N_0)=\Sigma_m q^{-4\rho(m)}$. Defining

$$\mathscr{P}_4=(C_q)_{N_4}(\tilde{C}_q)_{N_4} \tag{43}$$

we can express $\check{R}_q(x)$ in an explicit form:

$$\begin{aligned}
\check{R}_q(x)=\check{R}_q&+x\{(1-q^2)(1-q^{10})\mathbb{1}-(1+q^{18})\check{R}_q-q^{12}\check{R}_q^{-1}\\
&-(1+q^{-18})(1-q^{20})(1-q^{28})\mathscr{P}_4\}\\
&+x^2\{-q^{18}(1-q^2)(1-q^{10})\mathbb{1}+q^{18}\check{R}_q+q^{12}(1+q^{18})\check{R}_q^{-1}\\
&+(1+q^{-18})(1-q^{20})(1-q^{28})\mathscr{P}_4\}-x^3q^{30}\check{R}_q^{-1}.
\end{aligned} \tag{44}$$

Through a tedious calculation in terms of the Deguchi-Wadati-Akutsu algebra (Deguchi *et al* 1988), similar to that done in section 3, it can be checked that $\check{R}_q(x)$ given in (38), (40) and (44) does satisfy the spectrum-dependent Yang-Baxter equation (25).

The rational solution for the minimal representation N_0 of q-E_7 is the following limit:

$$\begin{aligned}\check{R}(u,\eta) &= \lim_{q\to 1} \check{R}_q(q^{2u/\eta})/(1-q^{2u/\eta})^3 \\ &= P\{\mathbb{1}+(2t+13.5\mathbb{1})\eta/u+R_2\eta^2/u^2+45P\eta^3/u^3\} \qquad (45)\end{aligned}$$

where t is given in (34) with the generators I_a belonging to E_7, and

$$\begin{aligned}R_2 &= 40.5\mathbb{1}+5P+18t+280P_4 \\ (P_4)_{m_1m_2,m_1'm_2'} &= (\mathscr{P}_4)_{m_1m_2,m_1'm_2'}\big|_{q=1} \\ &= (56)^{-1}(-1)^{\kappa(m_1)+\kappa(m_1')}\delta_{m_1\bar{m}_2}\delta_{m_1'\bar{m}_2'}.\end{aligned}$$

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Note added. The explicit expressions for e_0 for the minimal representations of the quantum E_6 and E_7 were found recently and the solutions $\check{R}_q(x)$ given in the present paper were proved to be correct. The details will be published elsewhere (Ma 1990). Besides, after submitting this paper I received two preprints: Chung and Koh 1990 and Zhang *et al* 1990, where the spectrum-dependent trigonometric solutions to the Yang-Baxter equation for the minimal representations of the quantum E_6, E_7 and E_8 were computed by the limit conditions independently but without the direct check.

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